MAT 219 (lecture 10)

§ 7.7 The matrix e^{At}

(~~§ 7.8 Repeated Eigenvalues (begin)~~) → Next Time

System of Linear DE

$$\begin{cases} x' = x + 9y \\ y' = -2x - 5y \end{cases} \quad \text{with} \quad \begin{matrix} x(0) = 3 \\ y(0) = 1 \end{matrix}$$

↓

Vector DE

$$\underline{x}' = \begin{bmatrix} 1 & 9 \\ -2 & -5 \end{bmatrix} \underline{x} \quad \text{with} \quad \underline{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

↓

Eigenvalues & Eigenvectors

$$\lambda = -2 \pm 3i \quad \text{has} \quad \underline{v} = \begin{bmatrix} 3 \\ -1 \pm i \end{bmatrix}$$

↓

General (Vector) Solution

$$\underline{x} = c_1 e^{-2t}\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix} \cos 3t - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin 3t \right) + c_2 e^{-2t}\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix} \sin 3t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos 3t \right)$$

↓

Plug in Initial Values

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \underline{x}(0) = c_1 \begin{bmatrix} 3 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightsquigarrow \boxed{\begin{matrix} c_1 = 1 \\ c_2 = 2 \end{matrix}}$$

↓

Solution to System

$$\begin{cases} x = 1 \cdot e^{-2t}(3 \cdot \cos 3t) + 2 \cdot e^{-2t}(3 \sin 3t) \\ y = 1 \cdot e^{-2t}(-\cos 3t - \sin 3t) + 2 \cdot e^{-2t}(-\sin 3t + \cos 3t) \end{cases}$$

What if there aren't enough eigenvalues / eigenvectors to get a full set of basic ("fundamental") solutions?

EX: $\begin{cases} x' = 5x - y \\ y' = 4x + y \end{cases} \rightsquigarrow \underline{x}' = \begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix} \underline{x}$

Eigenvalues: $\det \begin{bmatrix} 5-\lambda & -1 \\ 4 & 1-\lambda \end{bmatrix} = 0$

$$\lambda^2 - 6\lambda + 9 = 0$$

$$(\lambda - 3)^2 = 0 \qquad \boxed{\lambda = 3, 3}$$

(Maybe $\lambda = 3$ will have two "fundamentally different" eigenvect? (i.e. not multiples))

Eigenvectors

$$\begin{bmatrix} 5-3 & -1 \\ 4 & 1-3 \end{bmatrix} \underline{v} = \underline{0} \qquad \underline{v} = \begin{bmatrix} +1 \\ 2 \end{bmatrix}$$

(Bottom row gives $\underline{v} = \begin{bmatrix} +2 \\ 4 \end{bmatrix}$ ← same line)

- Only get one fundamental solution!

$$\underline{x}^{(1)} = e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \underline{x}^{(2)} = ??$$

To solve this, we need some new theory

Two lectures: Now §7.7: Fundamental Matrix and e^{At}

Next §7.8: Repeated Eigenvalues.

§7.7: Fundamental Matrix & e^{At}

Recall: Fundamental Matrix Ψ

(columns are the "fundamental solutions")

EX: Fundamental Matrix for $\underline{x}' = \begin{bmatrix} -1 & 6 \\ -2 & 6 \end{bmatrix} \underline{x}$

Eigenvalues $\lambda^2 - 5\lambda + 6 = 0$

$(\lambda-2)(\lambda-3) = 0$ $\boxed{\lambda = 2, 3}$

2-eigenvect: $\begin{bmatrix} -3 & 6 \\ -2 & 4 \end{bmatrix} \underline{v} = \underline{0} \rightsquigarrow \underline{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

3-eigenvect: $\begin{bmatrix} -4 & 6 \\ -2 & 3 \end{bmatrix} \underline{v} = \underline{0} \rightsquigarrow \underline{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

solution $\underline{x} = c_1 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$$= c_1 \begin{bmatrix} 2e^{2t} \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} 3e^{3t} \\ 2e^{3t} \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 2e^{2t} & 3e^{3t} \\ e^{2t} & 2e^{3t} \end{bmatrix}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

"Fundamental Matrix" Ψ

Note: $\underline{x} = \Psi \underline{c}$ is solution to $\underline{x}' = A\underline{x}$

$\updownarrow$ similar to DE

$y = e^{at} c$ is solution to $y' = ay$

Idea: Define e^{At} so that $\Psi \approx e^{At}$.

Baby example:

$\underline{x}' = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \underline{x}$ $\rightsquigarrow$

Eigenvalues $\lambda = 2, 3$

2-eigenvector

$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \underline{v} = \underline{0} \rightsquigarrow \underline{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

note: cannot do negative inverse of first row $[0\ 0]$

3-eigenvector

$\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \underline{v} = \underline{0} \rightsquigarrow \underline{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\Downarrow$ alternate soln.

$\begin{cases} x' = 2x \\ y' = 3x \end{cases}$

$\Downarrow$

$\begin{cases} x = c_1 e^{2t} \\ y = c_2 e^{3t} \end{cases}$

$\Downarrow$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 e^{2t} \\ c_2 e^{3t} \end{bmatrix}$ $\rightsquigarrow$

General Solution:

$\underline{x} = c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$= \boxed{\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Ψ

- So we want a definition that makes

$$e^{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} t} \approx \Psi = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix}$$

That doesn't seem too difficult....

Think of power series (aka Taylor series):

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots$$

Def: $e^A = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I \text{ "Identity matrix"}} + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \cdots$

Basic Computation: (fun!)

$$e^{\begin{bmatrix} 2t & 0 \\ 0 & 3t \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2t & 0 \\ 0 & 3t \end{bmatrix} + \frac{1}{2}\begin{bmatrix} (2t)^2 & 0 \\ 0 & (3t)^2 \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} 1 + (2t) + \frac{1}{2}(2t)^2 + \cdots & 0 \\ 0 & 1 + (3t) + \frac{1}{2}(3t)^2 + \cdots \end{bmatrix}$$

$$= \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix} \quad \text{(Wow!)}$$

- What about matrices that are not diagonal?
  - → Use eigenvalues and eigenvectors...

BASIC FACTS: If $A\underline{v} = \lambda \underline{v}$ then:

① $(At)\underline{v} = (\lambda t)\underline{v}$

② $A^2 \underline{v} = \lambda^2 \underline{v}$

More generally,

③ $A^n \underline{v} = \lambda^n \underline{v}$

④ $\underbrace{(I + A + \frac{1}{2}A^2 + \cdots)}_{= e^A}\underline{v} = \underbrace{(1 + \lambda + \frac{1}{2}\lambda^2 + \cdots)}_{= e^\lambda}\underline{v}$

→ So λ an eigenvalue of $A \Longleftrightarrow e^{\lambda t}$ eigenval of e^{At}

v an eigenvector of $A \Longleftrightarrow \underline{v}$ eigenvect of e^{At}

People who took MAT 260 now know that

$$e^{At} = \begin{bmatrix} | & | & \\ \underline{v}_1 & \underline{v}_2 & \cdots \\ | & | & \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & \ddots \end{bmatrix} \begin{bmatrix} | & | & \\ \underline{v}_1 & \underline{v}_2 & \cdots \\ | & | & \end{bmatrix}^{-1}$$

where $\underline{v}_1$ is λ_1-eigenvector of A

$\underline{v}_2$ is λ_2-eigenvector of A

(etc.)

We will return to the messy calculation of e^{At} using power series again later... (next lecture)

→ For now, we will merely note that

$$e^{At} \neq \Psi$$

because constants don't match:

$$\underline{x} = e^{At}\underline{d} \qquad \text{But} \qquad \underline{x} = \Psi \underline{c}$$

$$\underline{x}(0) = e^0 \underline{d} = \underline{d} \qquad \underline{x}(0) = \Psi(0)\underline{c}$$

$$\underline{d} = \Psi(0)\underline{c}$$

$$\boxed{\begin{array}{l} \text{→ Result: ① } \underline{c} = \Psi(0)^{-1} \cdot \underline{x}(0) \text{ (We knew this already)} \\ \text{② } e^{At} = \Psi(t) \cdot (\Psi(0))^{-1} \end{array}}$$

Note: e^{At} is also called "Φ" in the MAT 219 book

EX: $A = \begin{bmatrix} -1 & 6 \\ -2 & 6 \end{bmatrix}$ compute e^{At}.

(from previous work) $\Psi = \begin{bmatrix} 2e^{2t} & 3e^{3t} \\ e^{2t} & -2e^{3t} \end{bmatrix}$

So
$$e^{\begin{bmatrix} -1 & 6 \\ -2 & 6 \end{bmatrix} t} = \begin{bmatrix} 2e^{2t} & 3e^{3t} \\ e^{2t} & -2e^{3t} \end{bmatrix} \begin{bmatrix} 2\cancel{e^{0}} & 3\cancel{e^{0}} \\ 1\cancel{e^{0}} & -2\cancel{e^{0}} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 2e^{2t} & 3e^{3t} \\ e^{2t} & -2e^{3t} \end{bmatrix} \frac{1}{-4-3} \begin{bmatrix} -2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$= -\tfrac{1}{7} \begin{bmatrix} -4e^{2t} - 3e^{3t} & -6e^{2t} + 6e^{3t} \\ -2e^{2t} + 2e^{3t} & -3e^{2t} - 4e^{3t} \end{bmatrix}$$

(**Recall:** $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$)

e^{At} is useful because

$\underline{x}' = A\underline{x}$ with $\underline{x}(0) = \begin{bmatrix} \vdots \end{bmatrix}$

has solution $\underline{x} = e^{At} \cdot \underline{x}(0)$

(Just like $y' = ay$ with $y(0) = c$
has solution $y = e^{at} \cdot y(0)$)

Basic Formula:
$$e^{At} = \underbrace{\begin{bmatrix} | & | & \\ \underline{v}_1 & \underline{v}_2 & \cdots \\ | & | & \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & \ddots \end{bmatrix}}_{\Psi(t)} \cdot \underbrace{\begin{bmatrix} | & | & \\ \underline{v}_1 & \underline{v}_2 & \cdots \\ | & | & \end{bmatrix}^{-1}}_{\Psi(0)^{-1}}$$

EX: $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

Compute e^{At} and use it to solve
$\underline{x}' = A\underline{x}$ with $\underline{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Eigenvalues $\lambda^2 - 4\lambda + 3 = 0$
$(\lambda - 3)(\lambda - 1) = 0$ $\boxed{\lambda = 1, 3}$

1-Eigenvector $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \underline{v} = \underline{0} \rightsquigarrow \underline{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

3-Eigenvector $\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \underline{v} = \underline{0} \rightsquigarrow \underline{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$e^{At} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

Solution to I.V.P.

$$\underline{x} = e^{At} \underline{x}(0)$$

$$= \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{t} & 0 \\ 0 & e^{3t} \end{bmatrix}}_{\Psi} \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 3 \end{bmatrix}}_{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}$$

$$= \overbrace{\begin{bmatrix} e^{t} & e^{3t} \\ e^{t} & -e^{3t} \end{bmatrix}}^{\Psi} \cdot \overbrace{\frac{1}{-1-1} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}}^{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}$$

$$= \begin{bmatrix} e^{t} & e^{3t} \\ e^{t} & -e^{3t} \end{bmatrix} \left(-\tfrac{1}{2}\right) \begin{bmatrix} -2-3 \\ -2+3 \end{bmatrix}$$

$$= \left(-\tfrac{1}{2}\right) \begin{bmatrix} -5e^{t} + e^{3t} \\ -5e^{t} - e^{3t} \end{bmatrix}$$

EX: $A = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix}$

Compute e^{At} and use it to solve

$$\underline{x}' = A\underline{x} \quad \text{with} \quad \underline{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Eigenvalues $\lambda^2 - \lambda - 2 = 0$

$(\lambda-2)(\lambda+1) = 0$ $\quad \boxed{\lambda = -1, 2}$

(-1)-Eigenvect. $\begin{bmatrix} 4 & -1 \\ 4 & -1 \end{bmatrix}\underline{v} = \underline{0} \rightsquigarrow \underline{v} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

2-Eigenvect. $\begin{bmatrix} 1 & -1 \\ 4 & -4 \end{bmatrix}\underline{v} = \underline{0} \rightsquigarrow \underline{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$e^{At} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}^{-1}$$

Solution to I.V.P.

$$\underline{x} = e^{At} \cdot \underline{x}(0)$$

$$= \underbrace{\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix}}_{\Psi}\underbrace{\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}$$

$$= \overbrace{\begin{bmatrix} e^{-t} & e^{2t} \\ 4e^{-t} & e^{2t} \end{bmatrix}}^{\Psi} \cdot \overbrace{\frac{1}{1-4}\begin{bmatrix} 1 & -1 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}}^{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}$$

$$= \begin{bmatrix} e^{-t} & e^{2t} \\ 4e^{-t} & e^{2t} \end{bmatrix}\left(-\tfrac{1}{3}\right)\begin{bmatrix} 1 \\ -7 \end{bmatrix} = \left(-\tfrac{1}{3}\right)\begin{bmatrix} e^{-t} - 7e^{2t} \\ 4e^{-t} - 7e^{2t} \end{bmatrix}$$

The same thing works for $\mathbb{C}$-roots:

Recall: $\lambda = a \pm bi$ with $\underline{v} = \underline{\alpha} \pm \underline{\beta} i$

$$\Rightarrow \underline{x} = c_1 e^{at}(\underline{\alpha}\cos bt - \underline{\beta}\sin bt) + c_2 e^{at}(\underline{\alpha}\sin bt + \underline{\beta}\cos bt)$$

$$= \begin{bmatrix} e^{at}(\underline{\alpha}\cos bt - \underline{\beta}\sin bt) & e^{at}(\underline{\alpha}\sin bt + \underline{\beta}\cos bt) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} \underline{\alpha} & \underline{\beta} \end{bmatrix}\begin{bmatrix} e^{at}\cos bt & e^{at}\sin bt \\ -e^{at}\sin bt & e^{at}\cos bt \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} \underline{\alpha} & \underline{\beta} \end{bmatrix} e^{at} \underbrace{\begin{bmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{bmatrix}}_{\text{Rotation matrix (rotate by } bt)} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Result: If A has $\mathbb{C}$ eigenvalues $\lambda = a \pm bi$ with eigenvectors $\underline{v} = \underline{\alpha} \pm \underline{\beta} i$ then

$$e^{At} = \begin{bmatrix} \underline{\alpha} & \underline{\beta} \end{bmatrix} e^{at}\begin{bmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{bmatrix}\begin{bmatrix} \underline{\alpha} & \underline{\beta} \end{bmatrix}^{-1}$$

(Optional — If there is enough time)

EX: $A = \begin{bmatrix} 7 & -4 \\ 5 & -1 \end{bmatrix}$ (Example too long?)

Compute e^{At} and use it to solve

$\underline{x}' = A\underline{x}$ with $\underline{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Eigenvalues $\lambda^2 - 6\lambda + 13 = 0$

$(\lambda - 3)^2 + 4 = 0$ $\boxed{\lambda = 3 \pm 2i}$

$(3+2i)$-Eigenvect $\begin{bmatrix} 4-2i & -4 \\ 5 & -4-2i \end{bmatrix} \underline{v} = \underline{0} \leadsto \underline{v} = \begin{bmatrix} 2 \\ 2-i \end{bmatrix}$

$$e^{At} = \begin{bmatrix} 2 & 0 \\ 2 & -1 \end{bmatrix} e^{3t} \begin{bmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 2 & -1 \end{bmatrix}^{-1}$$

Solution to I.V.P.

$$\underline{x} = e^{At} \cdot \underline{x}(0)$$

$$= \underbrace{\begin{bmatrix} 2 & 0 \\ 2 & -1 \end{bmatrix} e^{3t} \begin{bmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{bmatrix}}_{\Phi} \underbrace{\begin{bmatrix} 2 & 0 \\ 2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 3 \end{bmatrix}}_{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}$$

$$= \overbrace{e^{3t} \begin{bmatrix} 2\cos 2t & 2\sin 2t \\ 2\cos 2t + \sin 2t & 2\sin 2t - \cos 2t \end{bmatrix}}^{\Phi} \overbrace{\left(\frac{1}{-2}\right) \begin{bmatrix} -1 & 0 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}}^{}$$

$$= e^{3t} \begin{bmatrix} 2\cos 2t & 2\sin 2t \\ 2\cos 2t + \sin 2t & 2\sin 2t - \cos 2t \end{bmatrix} \left(-\frac{1}{2}\right) \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

(EX continues)

$$= e^{3t} \cdot \left(-\frac{1}{2}\right) \begin{bmatrix} -2\cos 2t + 8\sin 2t \\ -6\cos 2t + 7\sin 2t \end{bmatrix}$$

→ Is this better than our old way to solve this? . . . maybe . . .

Anyway, it is always interesting to have a formula for the answer (instead of a method to solve).

Especially engineers always seem to want to have formulas. <sigh>